4.4 Semisimple Lie algebras and Lie groups.

Let as before $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$.

Definition 4.56 [(Semi) simple Lie algebras / groups]

1) $\mathfrak{g}$ is simple if

   a) $\mathfrak{g}$ is non-abelian;

   b) if $\mathfrak{h} \trianglelefteq \mathfrak{g}$ then either $\mathfrak{h}=\{0\}$ or $\mathfrak{h}=\mathfrak{g}$.

2) $\mathfrak{g}$ is semisimple if there are simple ideals $\mathfrak{h}_1, \ldots, \mathfrak{h}_n$ in $\mathfrak{g}$ such that as Lie algebras

$$\mathfrak{g}=\mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r .$$

This means that if $X=\sum_{i=1}^{r} X_i$ and $Y=\sum_{i=1}^{r} Y_i$ with $X_i, Y_i \in \mathfrak{h}_i$ then

$$[X,Y]=\sum_{i=1}^{r}[X_i,Y_i].$$

3) A connected Lie group is simple (respectively semisimple) if its

Lie algebra is .

Remark 4.57

An <u>abstract group</u> G is <u>simple</u> if it admits only two normal subgroups G itself and $\{e\}$.
We shall see that $SL(n,\mathbb{R})$ is a <u>simple Lie group</u>. However, it is not simple as an abstract group since $Z(SL(n,\mathbb{R})) = \{\pm 1\}$.

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The fundamental characterization of <u>semisimplicity</u> is given by the following,

Theorem 4.58

$\mathfrak{g}$ is <u>semisimple</u> if and only if $K_{\mathfrak{g}}$ is <u>non-degenerate</u>.

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Recall: a <u>symmetric bilinear form</u> $C : V \times V \longrightarrow \mathbb{K}$ is said to be <u>non-degenerate</u> if, setting
$\operatorname{rad}(C) := \{v \in V : C(v,v') = 0 \ \forall v' \in V\}$

it holds $\quad \operatorname{rad}(C) = \{0\}$.

$\_\_\_\_\_\_ \circ \_\_\_\_\_\_$

We will discuss one implication in the proof. For this we need the following: $\Longrightarrow$

Lemma 4.59

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \trianglelefteq \mathfrak{g}$ be <u>an ideal</u>. Then

$$\mathfrak{h}^{\perp} := \{ x \in \mathfrak{g} : \kappa_{\mathfrak{g}}(x,y) = 0 \ \forall y \in \mathfrak{h} \}$$

is an <u>ideal</u> as well.

Proof

Let $z \in \mathfrak{g}$, $x \in \mathfrak{h}^{\perp}$, $y \in \mathfrak{h}$. Then

$$\kappa_{\mathfrak{g}}(\operatorname{ad}(z)(x), y) = -\kappa_{\mathfrak{g}}(x, \operatorname{ad}(z)y) = 0$$

where we used Proposition 4.52 for the first equality and the assumption that $\mathfrak{h}$ is <u>an ideal</u> for the second one. $\square$

Proof of $(\Longrightarrow)$ in Theorem 4.58

Assume that $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_r$ with $\mathfrak{g}_i$ <u>simple</u> for $1 \leq i \leq r$.

Then $\forall\, x = \sum_{i=1}^{r} x_i$ it holds

$$\mathrm{ad}_{\mathfrak{g}}(x) = \begin{pmatrix} \mathrm{ad}_{\mathfrak{g}_1}(x_1) & & \\ & \ddots & \\ & & \mathrm{ad}_{\mathfrak{g}_r}(x_r) \end{pmatrix}.$$

← diagonal with blocks. follows from $[x,y] = \sum [x_i, y_i]$

Hence $K_{\mathfrak{g}}(x,y) = \sum_{i=1}^{r} K_{\mathfrak{g}_i}(x_i, y_i)$ $\forall x,y \in \mathfrak{g}$.

Therefore it is sufficient to discuss the case when $\mathfrak{g}$ is <u>simple</u> (Check it!).

Let $\mathfrak{g}^{\perp} = \mathrm{rad}(K_{\mathfrak{g}}) = \{ y \in \mathfrak{g} : K_{\mathfrak{g}}(x,y) = 0\ \forall x \in \mathfrak{g} \}$

Then $\mathfrak{g}^{\perp}$ is an ideal in $\mathfrak{g}$ by Lemma 4.59. Since $\mathfrak{g}$ is simple, either $\mathfrak{g}^{\perp} = \{0\}$ or $\mathfrak{g}^{\perp} = \mathfrak{g}$. If $\mathfrak{g}^{\perp} = \mathfrak{g}$ then $K_{\mathfrak{g}} \equiv 0$ hence $\mathfrak{g}$ is solvable by Cartan's Theorem 4.54, a contradiction to simplicity. Hence $\mathfrak{g}^{\perp} = \{0\}$, i.e., $K_{\mathfrak{g}}$ is <u>non-degenerate</u>. $\square$

Next we discuss a powerful way to produce families of <u>semisimple Lie algebras</u>:

Theorem 4.60

$\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ here

Let V be a $\mathbb{K}$-vector space endowed with an inner product $\langle , \rangle$. ~ positive definite

If $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a $\mathbb{K}$-subalgebra that is self-adjoint under $\langle , \rangle$ and such that $Z(\mathfrak{g}) = \{0\}$ then $K_\mathfrak{g}$ is non-degenerate and hence $\mathfrak{g}$ is semisimple.

$\stackrel{?}{\sim}$ Theorem 4.58

Note: for $A \in \mathfrak{gl}(V)$ we let $A^* \in \mathfrak{gl}(V)$ be defined by

$$\langle Av, w\rangle = \langle v, A^* w\rangle \quad \forall v, w \in V.$$

Then $\mathfrak{g}$ being self-adjoint means that $\forall A \in \mathfrak{g}$ it holds $A^* \in \mathfrak{g}$.

We can exploit Theorem 4.60 to produce a large family of examples of semisimple Lie algebras:

Example 4.61

1) $\mathfrak{sl}(n,\mathbb{R}) \subset \mathfrak{gl}(n,\mathbb{R})$ is invariant under $A \longmapsto {}^tA$ and $Z(\mathfrak{sl}(n,\mathbb{R})) = \{0\}$.

↪ adjoint with respect to standard inner product

2) $\mathfrak{sl}(n,\mathbb{C}) \subset \mathfrak{gl}(n,\mathbb{C})$ is invariant under $A \longmapsto {}^t\overline{A}$ and $Z(\mathfrak{sl}(n,\mathbb{C})) = \{0\}$.

↪ adjoint

3) For $p+q=n$,

$$\mathfrak{o}(p,q) := \{ X \in \mathfrak{gl}(n,\mathbb{R}) : {}^tX J_{p,q} + J_{p,q} X = 0 \}$$

where $J_{p,q} = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}$ is invariant under $X \longmapsto {}^tX$. Indeed from ${}^tX J_{p,q} + J_{p,q} X = 0$ we obtain by multiplying on the left and on the right by $J_{p,q}$ that $J_{p,q}\, {}^tX + X J_{p,q} = 0$ since $J_{p,q}^2 = \mathrm{Id}_n$.

One can also verify that $Z(\mathfrak{o}(p,q)) = \{0\}$.

We conclude with some hints towards the so called <u>Levi decomposition</u> of Lie groups.

Proposition 4.62
For any Lie algebra $\mathfrak{g}$ there is a unique maximal solvable ideal $\mathfrak{r} \subset \mathfrak{g}$. Moreover $\mathfrak{g}/\mathfrak{r}$ is semisimple.

Definition 4.63 [Radical]
The unique maximal solvable ideal from Proposition 4.62 is called the (solvable) radical of $\mathfrak{g}$.

For the proof of Proposition 4.62 we need the following:

Lemma 4.64

It is also the case that sum of ideals is an ideal.

If $\mathfrak{a}$ and $\mathfrak{b}$ are solvable ideals in a Lie algebra $\mathfrak{g}$ then $\mathfrak{a}+\mathfrak{b}$ is a solvable ideal.

The proof of Lemma 4.64 is left as an exercise. We discuss how to use it to prove Proposition 4.62.

Proof of Proposition 4.62

In order to prove existence and uniqueness of the maximal solvable ideal it suffices to exploit finite dimensionality and Lemma 4.64.

To show that $\mathfrak{g}/\mathfrak{r}$ is semisimple we first establish the following:

Claim: $\mathfrak{g}/\mathfrak{r}$ has no solvable ideals.

Indeed, let $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{r}$ be the canonical projection and let $\mathfrak{h} \subset \mathfrak{g}/\mathfrak{r}$ be a solvable ideal. Then $\mathfrak{a} := \pi^{-1}(\mathfrak{h}) \subseteq \mathfrak{g}$ is an ideal. Moreover $\pi(\mathfrak{a}) = \mathfrak{h}$ is solvable and $\operatorname{Ker} \pi|_{\mathfrak{a}}$ is solvable being contained in $\mathfrak{r}$. Therefore $\mathfrak{a}$ is solvable and hence $\mathfrak{a} \subseteq \mathfrak{r}$ and $\mathfrak{h} = \{0\}$.
$\rightsquigarrow$ Used Lemma 4.27 twice.

Next it is sufficient to observe that a Lie algebra with no non-trivial solvable ideals is semisimple.

Indeed, it is sufficient to prove that the Killing form $K_{\mathfrak{h}}$ of such Lie algebra is non-degenerate, as then the conclusion will follow from Theorem 4.58.

Assume that $K_{\mathfrak{h}}$ is not non-degenerate. Let $\mathfrak{h}^{\perp} := \mathrm{rad}(K_{\mathfrak{h}}) \neq \{0\}$. By Lemma 4.59 $\mathfrak{h}^{\perp}$ is an ideal. By Lemma 4.55 $K_{\mathfrak{h}^{\perp}} = K_{\mathfrak{h}}|_{\mathfrak{h}^{\perp} \times \mathfrak{h}^{\perp}}$. In particular, $K_{\mathfrak{h}^{\perp}} \equiv 0$. Hence by Theorem 4.54 $\mathfrak{h}^{\perp}$ is solvable. This contradicts the hypothesis thus $\mathfrak{h}^{\perp} = \{0\}$ and $K_{\mathfrak{h}}$ is non-degenerate. $\square$